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ON A HEIGHT OF SMOOTH FUNCTIONS WITH MULTIPLE COMPONENTS*

I. A. IKROMOV[1], A. SOLEEV[1]

Abstract. In the paper we consider estimates for height of the smooth phase function which is a product of few smooth functions. We prove that the height of functions is not large than sum of heights of its factors and show that it is strictly less than sum of heights of factors for some class of functions.

Keywords: oscillatory integral, Newton polygon, adapted coordinates system.

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1. Introduction

In the paper we consider relation between the height of factors and the hight the function which is product of the factors. A.N. Varchenko [11] solving the problem proposed by V.I. Arnold [1] proved the existence of so-called adapted coordinates system for the wide class of analytic functions of two-variables. Moreover, he states that such coordinates system exist for arbitrary analytic functions (see [3]). Later, an analog of Varchenko Theorem is proved for arbitrary analytic functions by D.H. Phong, J. A. Shtrum and E.M. Stein [9]. Note that for adapted coordinates system "distance" between the origin and the Newton's polygon defines the sharp behavior of the oscillatory integrals. A.N. Varchenko showed that for general functions of $n \geq 3$ variables such connection does not exist. Although H. Schultz [10] showed that for finite linear type smooth convex functions an analogical coordinates system exists. Moreover, one can show that an analog of such kind of coordinates system exists for arbitrary convex analytic functions [8].

Note that the behavior of the oscillatory integrals with smooth phase functions may be much more complicated than the behavior of oscillatory integrals with analytic phase functions (for behavior of oscillatory integrals with analytic phase functions (see. [2]). Nevertheless, an analog of adapted coordinates system exists for smooth functions of two variables [6]. Such coordinates systems allow to obtain "almost" sharp estimates for oscillatory integrals with smooth phase functions.

In this paper we consider smooth function which is a product of smooth factors. We obtain an estimate for the height of the function by using of sum of heights of its factors. Note that in some cases the height of function is strictly less than sum of heights of its factors.

Such kind of estimates are important in some problems connected to oscillatory integrals and also in investigation of so-called contact index of functions [7].

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[1] Samarkand State University, Samarkand, Uzbekistan,
e-mail: ikromov1@rambler.ru, asoleev@yandex.ru

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2. Some notation and definitions

Following [3] and [11] we introduce some notations. Let $Z_+ \subset \mathbb{R}_+ \subset \mathbb{R}$ be the set of all nonnegative integers, all nonnegative real numbers, and all real numbers respectively. Let $K \subset Z_+^n$. Newton polyhedron of a set K is defined by the convex hull in $\mathbb{R}_+^n$ of the set $\cup_{k\in K}(k+\mathbb{R}_+^n)$.

Let f be a smooth function in a neighborhood of zero. Consider the Taylor series of this function centered at the origin

$$f_x \approx \sum_{k\in Z_+^n} c_k x^k, \quad c_n \in \mathbb{R}.$$

Let us write $supp(f_x) = \{k \in Z_+^n \setminus \{0\} : c_k \neq 0\}$.

Newton's polyhedron of a Taylor series of f is defined by Newton's polyhedron of the set $supp(f_x)$. For practical construction of the Newton's polyhedra see [4].

Let us specify a coordinate system in $\mathbb{R}^n$ and denote by f_x the Taylor series of the function f in this coordinates system. Let us denote by d a coordinate of intersection of the straight line $x_1 = \cdots = x_n = d$, $d \in \mathbb{R}$, and the boundary of the Newton's polyhedron. This number will be called a distance between Newton's polyhedron and the origin. The distance is denoted by $d(x)$. A principal face is the face of minimal dimension containing the point $(d(x), \ldots, d(x))$.

Let f be as above and $x = (x_1, \ldots, x_n)$ be local coordinates system at the origin in $\mathbb{R}^n$. Let f_x be Taylor series of f centered at the origin and $d(x)$ be the distance between the origin and Newton's polyhedron $N(f_x)$. Let us write $h(f) = sup\{d(x)\}$, where "supremum" is taken over the set of all local smooth coordinates systems x at the origin. The number $h(f)$ is called to be a height of the function f [11]. The local coordinates system is called to be adapted to the phase function f if $h(f) = d(x)$.

As noted before the sharp behavior of two-dimensional oscillatory integrals defines by the height of the phase function. But, the polynomial function constructed by A.N. Varchenko shows that in general for $n \geq 3$ the "height" is useless to define the sharp behavior of oscillatory integrals. On the other hand the height gives the sharp behavior for oscillatory integrals with smooth finite linear type convex functions [10] or with arbitrary convex analytic phase functions [8].

3. The main results

Let f_1, f_2, …, f_k be smooth functions in $\mathbb{R}^2$, satisfying the conditions: $f_l(0) = 0$, $\nabla f_l(0) = 0\,(l = 1, \ldots, k)$. We assume that, for each l function f_l has a finite type at the origin. This means that for some positive integer $N \geq 2$ it holds the relation $d^N f_l(0) \neq 0$.

Then the height of the function f_l is well-defined [6] and also we can define a height for the function

$$f(x) := f_1(x) \ldots f_k(x).$$

Theorem 3.1. *The following estimate*

$$h(f) \leq h(f_1) + \cdots + h(f_k) \tag{1}$$

holds.

Remark 3.1. *Note that in general the relation 1 is strict inequality. We can consider the simple example: $f(x) = f_1(x)f_2(x)$, where $f_1(x) = x_1^2$ and $f_2(x) = x_2^2$. We have $h(f_1) = h(f_2) = h(f) = 2$ for this example.*

Proof. As we noted before the height of the function is defined by the distance between the origin and the Newton's polygon constructed in adapted coordinates system. But, for the proof of the Theorem 3.1 we use another property of the height of smooth functions which proved in [7]. The height can be defined by the following relation

$$h(f) = \inf\{p : \text{ there exists } U \text{ such, that} \int_U \frac{dx}{|f(x)|^{1/p}} < +\infty\}. \tag{2}$$

The relation (2) follows from the results proved in the papers [11] for analytic functions without multiple components and [9] for arbitrary analytic functions.

Let α be a fixed number belonging to the interval $(0,1)$ and d be a number defined by

$$d := \frac{1}{\alpha}\sum_{l=1}^{k} h_l,$$

where $h_l := h(f_l)$, $h := h(f)$.

We show that the inequality $d \geq h$ holds. Indeed, by using the generalized Hölder's inequality we get

$$\int_U \frac{dx}{|f(x)|^{1/d}} \leq \prod_{l=1}^{k}\left(\int_U \frac{dx}{|f(x)|^{\alpha/h_l}}\right)^{q_l}, \tag{3}$$

where

$$q_l := \frac{1}{h_l}\sum_{j=1}^{k} h_j.$$

Note that

$$\sum_{l=1}^{k}\frac{1}{q_l} = 1.$$

Since it holds the inequality $h_l < h_l/\alpha$ for any l, then the left hand side of the inequality (3) is a finite number. Therefore we have the convergent integrals in right hand side of the inequality (3). Now due to the relations (2) we have: $d \geq h(f)$. Since α is any fixed number from the interval $(0,1)$, we obtain the desired inequality.

□

Let f be a function defined by $f = f_1 f_2$, where f_1, f_2 are smooth functions in $\mathbb{R}^2$, satisfying the conditions: $f_l(0) = 0$, $\nabla f_l(0) = 0\,(l = 1,\, 2)$.

Theorem 3.2. *If local adapted coordinates system to both functions f_1, f_2 simultaneously does not exist, then the following inequality*

$$h(f) < h(f_1) + h(f_2) \tag{4}$$

holds.

Proof. Due to results of the papers [11], [9], [6] there exists adapted to f coordinates system. Let $(x_1,\, x_2)$ be some adapted to f coordinates system. By conditions of the Theorem 3.2 the coordinates are not adapted at least for one of the factors. We suppose that the principal face of Newton polygon of the function f lies on the line given by the equation $\kappa_1 t_1 + \kappa_2 t_2 = 1$ in addition without loss of generality we may assume $\kappa_2 \geq \kappa_1$.

First, we suppose that $\kappa_1 > 0$. Then f, f_1, f_2 can be written in the form

$$f(x_1,x_2) = P(x_1,x_2) + R(x_1,x_2),\; f_l(x_1,x_2) = P_l(x_1,x_2) + R_l(x_1,x_2),\; (l = 1,2), \tag{5}$$

where P is a weighted homogeneous polynomial of degree one with respect to weights (κ_1, κ_2), which corresponds to the principal edge of Newton's polygon of the function f, and P_1, P_2 are weighted homogeneous polynomials with respect to the same weights of degree α_1, α_2 respectively, defined by Taylor series of the functions f_1, f_2. Since P is a weighted homogeneous polynomial of degree one and $P = P_1P_2$ then we have $\alpha_1 + \alpha_2 = 1$.

Note that some of the polynomials P_1, P_2 may not correspond to the principal face of Newton's polygon of the associated function.

If the principal face of Newton's polygon of the function f_1 does not lie on the line defined by $\{(t_1, t_2) : \kappa_1 t_1 + \kappa_2 t_2 = \alpha_1 < 1\}$, then we have $d_1 > \frac{\alpha_1}{|\kappa|}$ for the distance d_1 between the origin and Newton's polygon. Also we have the inequality $d_2 \geq \frac{\alpha_2}{|\kappa|}$. Consequently,

$$h(f_1) + h(f_2) \geq d_1 + d_2 > \frac{\alpha_1 + \alpha_2}{|\kappa|} = \frac{1}{|\kappa|} = h.$$

The last inequality holds for the case when the principal face of Newton's polygon of the function f_2 does not lie on the line $\{(t_1, t_2) : \kappa_1 t_1 + \kappa_2 t_2 = \alpha_2 < 1\}$.

Thus the inequality (3.1) stated in the Theorem 3.2 is proved in the considered case.

The remaining case the principal faces of Newton's polygons of the functions f_1 and f_2 lie on the lines $\{(t_1, t_2) : \kappa_1 t_1 + \kappa_2 t_2 = \alpha_1\}$ and $\{(t_1, t_2) : \kappa_1 t_1 + \kappa_2 t_2 = \alpha_2\}$ respectively.

Then for the distance between the origin and Newton polygon we have: $d_l = \frac{\alpha_l}{|\kappa|}$ $(l = 1, 2)$.

Note that the coordinates system does not adapted at least for one of the functions. Then without loss of generality we may assume that $h(f_1) > d_1$, in addition $h(f_2) \geq d_2$. Thus, we get the inequality

$$h(f_1) + h(f_2) > d_1 + d_2 = \frac{\alpha_1 + \alpha_2}{|\kappa|} = \frac{1}{|\kappa|} = h.$$

The last inequality finishes a proof of the Theorem 3.2 for the case when the principal face of the Newton polygon of the function f is a compact set in adapted coordinates system.

Now, suppose that the principal face of Newton's polygon of the function f is unbounded and it lies on the line given by the equation $\kappa_2 t_2 = 1$. Denote by (k_1^1, k_2^1) and (k_1^2, k_2^2) the points with minimal k_2^1 and k_2^2 living on the Newton's diagrams of the functions f_1 and f_2 respectively. Then obviously we have $k_2^l \leq d(f_l) (l = 1, 2)$, and also $(k_1^1 + k_1^2, k_2^1 + k_2^2) \in N(f)$.

Since

$$N(f) \subset \{(t_1, t_2) : t_2 \geq \frac{1}{\kappa_2}\},$$

then $k_2^1 + k_2^2 \geq 1/\kappa_2$. Consequently,

$$h(f) = \frac{1}{\kappa_2} \leq k_2^1 + k_2^2 \leq d(f_1) + d(f_2) < h(f_1) + h(f_2).$$

Note that the coordinates system are not adapted at least for one of the functions f_1, f_2. Therefore at least for one of that functions the principal face does not lie on line given by an analogical equation. Because, if the principal face of Newton's polygon of the function is unbounded then the coordinates system are adapted.

Suppose the principal face of the function f_1 is unbounded. Then we know that the coordinates system are adapted to f_1 (see [11] and also [6]). Therefore by our condition the coordinates system are not adapted to the function f_2. Then it is easy to see that the following inequality

$$h(f) \leq h(f_1) + d_2 < h(f_1) + h(f_2)$$

holds.

$\square$

Corollary 3.1. *If the adapted coordinates system simultaneously to the functions $f_1, f_2, \dots, f_k$ do not exist then the following inequality*

$$h(f_1, f_2, \dots, f_k) < h(f_1) + h(f_2) + \dots + h(f_k) \tag{6}$$

holds.

Theorem 3.3. *If local coordinates system do not adapted to the function f then the following inequality*

$$h(x_1^m x_2^n f) < h(f) + \max\{m, n\}. \tag{7}$$

holds.

Remark 3.2. *Note that the Theorem 3.3 does not follow from the Theorem 3.2.*

Proof. First, we consider the case $m = 0$ and $n \geq 1$. Then the Newton's polygon of the function $x_2^n f$ coincides with the shifted Newton's polygon of the function f by the vector $(0, n)$ e.g. $N(x_2^n f) = (0, n) + N(f)$. Since the given coordinates system do not adapted to f, then due to [6] the principal face of Newton's polygon is a compact edge. Suppose that the principal face lies on the line $L := \{(t_1, t_2) : t_1\kappa_1 + t_2\kappa_2 = 1\}$.

If the principal face of Newton's polygon $N(x_2^n f)$ does not lie on the line $L_1 := (0, n) + L$ then it is easy to see that the coordinates system are adapted to $x_2^n f$, therefore we have the inequality $m(P) \leq d_x < n + d_x$ for the weighted homogeneous polynomial P associated to the principal face of the Newton's polygon of the function $x_2^n f$ (see [9]).

Assume the principal face of the Newton's polygon $N(x_2^n f)$ lies on the line $L' := \{(t_1, t_2) : t_1\kappa_1' + t_2\kappa_2' = 1 + n\kappa_2'\}$. Then we have $\kappa_1' > 0$ $\kappa_1' + \kappa_2' \geq \kappa_1 + \kappa_2$. Consequently,

$$h(x_2^n f) = d(x_2^n f) = \frac{1 + n\kappa_2'}{\kappa_1' + \kappa_2'} < \frac{1}{\kappa_1' + \kappa_2'} + n < \frac{1}{\kappa_1 + \kappa_2} + n < h(f) + n.$$

Thus, the coordinates system are adapted to the function $x_2^n f$, and we have the estimate

$$d(x_1^n f) < d(f) + n < h(f) + n.$$

It remains to consider the case when the given coordinates system do not adapted to the function $x_2^n f$. Then the principal face of $N(x_2^n f)$ lies on the line $L := \{(t_1, t_2) : t_1\kappa_1 + t_2\kappa_2 = 1 + n\kappa_2\}$.

First, we consider the case $\kappa_1 \geq \kappa_2$. Then due to [11], (see [6] for the smooth case) there exists an analytic (smooth function $\varphi(x_2)$ such that the new coordinates system given by change of variables $x_1 - \varphi(x_2) \mapsto x_1$, $x_2 \mapsto x_2$ are adapted to the function$x_2^n f$. In addition the form of the function is invariant up to such change of variables. Thus we reduce our problem to the considered case.

Remark 3.3. *Note that the required estimate holds without any condition on the function f in the case $\kappa_1 \geq \kappa_2$.*

Now we consider the case $\kappa_1 \leq \kappa_2$. Then due to [11](for the case of smooth functions see [6]) there exists an analytic (smooth) function $\varphi(x_1)$ such that the new coordinates system given by change of variables $x_1 \mapsto x_1$, $x_2 - \varphi(x_1) \mapsto x_2$ are adapted to the function $x_2^n f$. Moreover, $\varphi(x_1)$ can be developed to the formal Taylor series

$$\varphi(x_1) = c_1 x_1^{k_1} + \dots.$$

If the principal face of Newton's polygon of the new function $f_1(x_1, x_2) := (x_2 - \varphi(x_1))^n f(x_1, x_2 - \varphi(x_1))$ lies on the line $t_1\kappa_1' + t_2\kappa_2' = 1$, then $k_1 < \kappa_2'/\kappa_1'$ [6].

Now, we write the Taylor series for the functions $(x_2-\varphi(x_1))^n$ and $f(x_1, x_2-\varphi(x_1))$ by using the weights (k_1', κ_2') as a result we have

$$(x_2-\varphi(x_1))^n = c_1^n x_1^{nk_1} + \ldots, \quad f(x_1, x_2-\varphi(x_1)) = p(x_1, x_2) + \ldots,$$

where $p(x_1, x_2)$ is a weighted homogeneous polynomial function with degree α. Then we get

$$h(f_1) = \frac{1}{\kappa_1' + \kappa_2'} = \frac{nk_1\kappa_1' + \alpha}{\kappa_1' + \kappa_2'} < \frac{n\kappa_2'}{\kappa_1' + \kappa_2'} + \frac{\alpha}{\kappa_1' + \kappa_2'} \leq n + h(f).$$

The case $m = 0$ can be considered by the analogical arguments. Thus in the case $m = 0$ and $n = 0$ the statement of the Theorem 3.3 is proved.

Now, we consider the case $m \geq 1$ and $n \geq 1$. Suppose, $m = n \geq 1$. The general case follows from that case by using the Theorem 3.1.

Let the given coordinates system are adapted to the function $f_1(x_1, x_2) := (x_2x_1)^n f(x_1, x_2)$. So they are not adapted to the function $f(x_1, x_2)$, then we get

$$h(f_1) = d(f_1) = n + d(f) < n + h(f).$$

Finally, we suppose that the given coordinates system do not adapted to the function $f_1(x_1, x_2)$ and the principal face of the Newton's polygon of the function coincides with the shifted to (n, n) principal face of the $f(x_1, x_2)$. Therefore, it lies on the line

$$L = (n, n) + \{(t_1, t_2) : t_1\kappa_1 + t_2\kappa_2 = 1\}.$$

Without loss of generality, we may assume that $\kappa_2 \geq \kappa_1$. Then κ_2/κ_1 is a positive integer number. Consequently, due to [11](for smooth case see [6]) there exists an analytic (smooth) function $\varphi(x_1)$ such that the new coordinates system given by change of variables $x_1 \mapsto x_1$, $x_2 - \varphi(x_1) \mapsto x_2$ are adapted to the function $(x_1x_2)^n f$. We write the Taylor (formal Taylor) series for the function $\varphi(x_1)$ and have

$$\varphi(x_1) = c_1 x_1^{k_1} + \ldots,$$

If the principal face of the Newton's polygon of the function

$$f_1(x_1, x_2) := (x_1(x_2 - \varphi(x_1)))^n f(x_1, x_2 - \varphi(x_1))$$

lies on the line $t_1\kappa_1' + t_2\kappa_2' = 1$ then $k_1 < \kappa_2'/\kappa_1'$ (see [6]).

Now, we write the Taylor development for the functions $(x_1(x_2-\varphi(x_1)))^n$ and $f(x_1, x_2-\varphi(x_1))$ corresponding to the weights (k_1', κ_2') and have

$$(x_1(x_2-\varphi(x_1)))^n = c_1^n x_1^{nk_1+n} + \ldots, \quad f(x_1, x_2-\varphi(x_1)) = p(x_1, x_2) + \ldots,$$

where $p(x_1, x_2)$ is a weighted homogeneous polynomial of degree α. Then we have

$$h(f_1) = \frac{1}{\kappa_1' + \kappa_2'} = \frac{(nk_1 + n)\kappa_1' + \alpha}{\kappa_1' + \kappa_2'} < \frac{n\kappa_2' + n\kappa_1'}{\kappa_1' + \kappa_2'} + \frac{\alpha}{\kappa_1' + \kappa_2'} \leq n + h(f).$$

Therefore, we get the required estimate for the case $m = n \geq 1$. The general case follows from the obtained results by using the Corollary 3.

□

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Isroil Ikromov graduated from Lomonosov Moscow State University in 1985. He got Ph.D. degree from Moscow State University in 1989 and Doctor of Physics and Mathematics degree from Institute of Mathematics of Academy of Sciences of Uzbekistan (Tashkent) in 1999. Presently he is a Professor of Department of Mechanics and Mathematics of Samarkand State University. His scientific researches include estimations of oscillatory integrals.

Akhmadjon Soleev was born in 1951 in Samarkand region. He graduated from Samarkand State University in 1973. From 1994 to 1998 he is a head of the chair in Samarkand State University. Since 1998 he is a vice-rector on Sciences in Samarkand State University. His scientific researches include singularity theory and algebraic geometry.